

Seshadri constants via Okounkov functions and the Segre-Harbourne-Gimigliano-Hirschowitz Conjecture

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April 2, 2013

Abstract

In this paper we relate the SHGH Conjecture to the rationality of one-point Seshadri constants on blow ups of the projective plane, and explain how rationality of Seshadri constants can be tested with the help of functions on Newton–Okounkov bodies.

Keywords Nagata Conjecture, SHGH Conjecture, Seshadri constants, Okounkov bodies

Mathematics Subject Classification (2000) MSC 14C20

1 Introduction

Nagata’s conjecture and its generalizations have been a central problem in the theory of surfaces for many years, and much work has been done towards verifying them [19], [8], [13], [23], [9]. In this paper we open a new line of attack in which we relate Nagata-type statements to the rationality of one-point Seshadri constants and invariants of functions on Newton–Okounkov bodies. We obtain as a consequence of our approach some evidence that certain Nagata-type questions might be false.

Seshadri constants were first introduced by Demailly in the course of his work on Fujita’s conjecture [10] in the late 80’s and have been the object of considerable interest ever since. Recall that given a smooth projective variety X and a nef line bundle L on X , the Seshadri constant of L at a point $x \in X$ is the real number

$$\varepsilon(L; x) =_{\text{def}} \inf_C \frac{L \cdot C}{\text{mult}_x C} , \quad (1)$$

where the infimum is taken over all irreducible curves passing through x . An intriguing and notoriously difficult problem about Seshadri constants on surfaces is the question whether these invariants are rational numbers, see [17, Remark 5.1.13]. It follows quickly from their definition that if a Seshadri constant is irrational then it must be $\varepsilon(L; x) = \sqrt{L^2}$, see e.g. [3, Theorem 2.1.5]. It is also known that Seshadri constants of a fixed line bundle L , take their maximal value on a subset in X which is a complement of at most countably many Zariski closed proper subsets of X .

*During this project Alex Küronya was partially supported by DFG-Forschergruppe 790 “Classification of Algebraic Surfaces and Compact Complex Manifolds”, and the OTKA Grants 77476 and 81203 by the Hungarian Academy of Sciences.

[†]Szemberg research was partially supported by NCN grant UMO-2011/01/B/ST1/04875

We denote this maximum by $\varepsilon(L, 1)$. Similar notation is used for multi-point Seshadri constants, see [3, Definition 1.9]. In particular, if $\varepsilon(L; x) = \sqrt{L^2}$ at some point the same holds in a very general point on X and $\varepsilon(L; 1) = \sqrt{L^2}$.

From a slightly different point of view, Seshadri constants reveal information on the structure of the nef cone on the blow-up of X at x , hence their study is closely related to our attempts to understand Mori cones of surfaces.

An even older problem concerning linear series on algebraic surfaces is the conjecture formulated by Beniamino Segre in 1961 and rediscovered, made more precise and reformulated by Harbourne 1986, Gimigliano 1987 and Hirschowitz 1988. (See [8] for a very nice account on this development and related subjects.) In particular it is known, [8, Remark 5.12] that the SHGH Conjecture implies the Nagata Conjecture. We now recall this conjecture, using by Gimigliano's formulation, which will be the most convenient form for us [12, Conjecture 3.3].

SHGH Conjecture. *Let X be the blow up of the projective plane \mathbb{P}^2 in s general points with exceptional divisors E_1, \dots, E_s . Let H denote the pullback to X of the hyperplane bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ on \mathbb{P}^2 . Let the integers $d, m_1 \geq \dots \geq m_s \geq -1$ with $d \geq m_1 + m_2 + m_3$ be given. Then the line bundle*

$$dH - \sum_{i=1}^s m_i E_i$$

is non-special.

The main result of this note is the following somewhat unexpected relation between the SHGH Conjecture and the rationality problem for Seshadri constants.

Theorem 1.1. *Let $s \geq 9$ be an integer for which the SHGH Conjecture holds true. Let X be the blow up of the projective plane \mathbb{P}^2 in s general points. Then*

- a) *either there exists on X an ample line bundle whose Seshadri constant at a very general point is irrational;*
- b) *or the SHGH Conjecture fails for $s + 1$ points.*

Note that it is known that the SHGH conjecture holds true for $s \leq 9$, [8, Theorem 5.1]. It is also known that Seshadri constants of ample line bundles on del Pezzo surfaces (i.e. for $s \leq 8$) are rational, see [22, Theorem 1.6]. In any case, the statement of the Theorem is interesting (and non-empty) for $s = 9$. (See the challenge at the end of the article.)

Corollary 1.2. *If all one-point Seshadri constants on the blow-up of \mathbb{P}^2 in nine general points are rational, then the SHGH conjecture fails for ten points.*

An interesting feature of our proof is that the role played by the general position of the points at which we blow up becomes clear.

In a different direction, we study the connection between functions on Newton–Okounkov bodies defined by orders of vanishing, and Seshadri-type invariants. Our main result along these lines is the following.

Theorem 1.3. *Let X be a smooth projective surface, Y_\bullet an admissible flag, L a big line bundle on X , and let $P \in X$ be an arbitrary point.*

$$\text{If } \max_{x \in \Delta_{Y_\bullet}(L)} \phi_{\text{ord}_P}(x) \in \mathbb{Q} \text{ then } \varepsilon(L, P) \in \mathbb{Q} .$$

Acknowledgements. We thank Cristiano Bocci, Sébastien Boucksom and Patrick Graf for helpful discussions. Part of this work was done while the second author was visiting the Uniwersytet Pedagogiczny in Cracow. We would like to thank the Uniwersytet Pedagogiczny for the excellent working conditions.

2 Rationality of one point Seshadri constants and the SHGH Conjecture

In this section we prove Theorem 1.1: we start with notation and preliminary lemmas. Let $f : X \rightarrow \mathbb{P}^2$ be the blow up of \mathbb{P}^2 at $s \geq 9$ general points P_1, \dots, P_s with exceptional divisors E_1, \dots, E_s . We denote as usual by $H = f^*(\mathcal{O}_{\mathbb{P}^2}(1))$ the pull back of the hyperplane bundle and we let $\mathbb{E} = E_1 + \dots + E_s$ be the sum of exceptional divisors. We consider the blow up $g : Y \rightarrow X$ of X at P with exceptional divisor F . Whilst The following result is well known, we include it for the lack of a proper reference.

Lemma 2.1. *If there exists a curve $C \subset X$ in the linear system $dH - \sum_{i=1}^s m_i E_i$ computing Seshadri constant of a \mathbb{Q} -line bundle $L = H - \alpha \mathbb{E}$, then there exists a divisor Γ with $\text{mult}_{P_1} \Gamma = \dots = \text{mult}_{P_s} \Gamma = M$ computing Seshadri constant of L at P , i.e.*

$$\frac{L \cdot \Gamma}{\text{mult}_P \Gamma} = \frac{L \cdot C}{\text{mult}_P C} = \varepsilon(L; P).$$

Proof. Since the points P_1, \dots, P_s are general, there exist curves

$$C_\sigma = dH - \sum_{j=1}^s m_{\sigma(j)} E_j$$

for all permutations $\sigma \in \Sigma_s$. Since the point P is general, we may take all these curves to have the same multiplicity m at P . Summing over a cycle σ of length s in Σ_s , we obtain a divisor

$$\Gamma = \sum_{i=1}^s C_{\sigma^i} = sdH - \sum_{i=1}^s \sum_{j=1}^s m_{\sigma^i(j)} E_j = sdH - M\mathbb{E},$$

with $M = m_1 + \dots + m_s$. Note that the multiplicity of Γ at P equals sm . Taking the Seshadri quotient for Γ we have

$$\frac{L \cdot \Gamma}{sm} = \frac{sd - \alpha sM}{sm} = \frac{d - \alpha M}{m} = \varepsilon(L; P)$$

hence Γ satisfies the assertions of the Lemma. \square

The following auxiliary Lemma will be used in the proof of Theorem 1.1. We postpone its proof to the end of this section.

Lemma 2.2. *Let $s \geq 9$ be an integer. The function*

$$f(\delta) = (2\sqrt{s+1} - s)\sqrt{1 - s\delta^2} + s(1 - \sqrt{s+1})\delta + s - 2 \quad (2)$$

is non-negative for δ satisfying

$$\frac{1}{\sqrt{s+1}} < \delta < \frac{1}{\sqrt{s}}. \quad (3)$$

Proof of Theorem 1.1. Let δ be a rational number satisfying (3). Note that the SHGH Conjecture implies the Nagata Conjecture [8, Remark 5.12] so that

$$\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); s) = \frac{1}{\sqrt{s}}$$

and hence the \mathbb{Q} -divisor $L = H - \delta E$ is ample. If $\varepsilon(L; 1)$ is irrational, then we are done.

So we proceed assuming that $\varepsilon(L; 1)$ is rational and that it is not equal to $\sqrt{L^2}$ (this can be achieved changing δ a little bit if necessary). In particular, by Lemma 2.1 for a general point $P \in X$ Seshadri constant $\varepsilon(L; P)$ there is a divisor $\Gamma \subset \mathbb{P}^2$ of degree γ with $M = \text{mult}_{P_1} \Gamma = \dots = \text{mult}_{P_s} \Gamma$ and $m = \text{mult}_P \Gamma$ whose proper transform $\tilde{\Gamma}$ on X computes the Seshadri constant

$$\varepsilon(L; P) = \frac{L \cdot \tilde{\Gamma}}{m} = \frac{\gamma - \delta s M}{m} < \sqrt{1 - s\delta^2}.$$

This gives an upper bound on γ

$$\gamma < m\sqrt{1 - s\delta^2} + \delta s M. \quad (4)$$

We need to prove that statement b) in Theorem 1.1 holds. Suppose not: the SHGH Conjecture then holds for $s+1$ points in \mathbb{P}^2 . The Nagata Conjecture then also holds for $s+1$ points and this gives a lower bound for γ , since for Γ we must have that

$$\frac{\gamma}{sM + m} \geq \frac{1}{\sqrt{s+1}}. \quad (5)$$

We now claim that

$$\gamma \geq 2M + m. \quad (6)$$

Suppose not. We then have that

$$\gamma < 2M + m. \quad (7)$$

The real numbers

$$a := \frac{2\sqrt{s+1} - s}{2 - \delta s} \quad \text{and} \quad b := \frac{s - \delta s\sqrt{s+1}}{2 - \delta s}$$

are positive. Multiplying (4) by a and (7) by b and adding we obtain

$$sM + m \leq \gamma\sqrt{s+1} < sM + (b + a\sqrt{1 - s\delta^2})m,$$

where the first inequality follows from (5). Subtracting sM in the left and in the right term and dividing by m we obtain

$$1 < b + a\sqrt{1 - s\delta^2}.$$

Plugging in the definition of a and b and rearranging terms we obtain that

$$(2\sqrt{s+1} - s)\sqrt{1 - s\delta^2} + s - \delta s\sqrt{s+1} < 2 - \delta s,$$

which contradicts Lemma 2.2. Hence (6) holds.

It follows now from the SHGH conjecture for $s + 1$ points (in the form stated in the introduction) that the linear system

$$\gamma H - M\mathbb{E} - mF$$

on Y is non-special. Indeed the condition $\gamma \geq 2M + m$ is (6) and the condition $\gamma \geq 3M$ is satisfied since $\frac{\gamma}{sM} > \frac{1}{\sqrt{s}}$ (because the Nagata Conjecture holds for s by hypothesis) and because we have assumed that $s \geq 9$. This system is also non-empty because the proper transform of Γ under g is its member. Thus by a standard dimension count

$$0 \leq \gamma(\gamma + 3) - sM(M + 1) - m(m + 1).$$

The upper bound on γ (4) together with the above inequality yields

$$0 \leq (s\delta M + m\sqrt{1 - s\delta^2})(s\delta M + m\sqrt{1 - s\delta^2} + 3) - m^2 - m - sM - sM^2. \quad (8)$$

Note that the quadratic term in (8) is a negative semi-definite form

$$(s^2\delta^2 - s)M^2 + 2s\delta\sqrt{1 - s\delta^2}Mm - s\delta^2m^2.$$

Indeed, the restrictions on δ made in (3) imply that the term at M^2 is negative. The determinant of the associated symmetric matrix vanishes. These two conditions imply together that the form is negative semi-definite. In particular this term of (8) is non-positive. The linear part in turn is

$$(3s\delta - s)M + (3\sqrt{1 - s\delta^2} - 1)m,$$

which is easily seen to be negative. This provides the desired contradiction and finishes the proof of the Theorem. \square

Remark 2.3. As it is well known, Nagata's conjecture can be interpreted in terms of the nef and Mori cones of the blow-up X of \mathbb{P}^2 at s general points. More precisely, consider the following question: for what $t \geq 0$ does the ray $H - t\mathbb{E}$ meet the boundary of the nef cone? The conjecture predicts that this ray should intersect the boundaries of the nef cone and the effective cone at the same time.

Considering the Zariski chamber structure of X (see [5]), we see that this is equivalent to requiring that $H - t\mathbb{E}$ crosses exactly one Zariski chamber (the nef cone itself). Surprisingly, it is easy to prove that $H - t\mathbb{E}$ cannot cross more than two chambers.

Proposition 2.4. *Let $f : X \rightarrow \mathbb{P}^2$ be the blow up of \mathbb{P}^2 in s general points with exceptional divisors E_1, \dots, E_s . Let H be the pull-back of the hyperplane bundle and $\mathbb{E} = E_1 + \dots + E_s$. The ray $R = H - t\mathbb{E}$ meets at most two Zariski chambers on X .*

Proof. If $\varepsilon = \varepsilon(\mathcal{O}_{\mathbb{P}^2}(1); s) = \frac{1}{\sqrt{s}}$ i.e. this multi-point Seshadri constant is maximal, then the ray crosses only the nef cone.

If ε is submaximal, then there is a curve $C = dH - \sum m_i E_i$ computing this Seshadri constant, i.e. $\varepsilon = \frac{d}{\sum m_i}$.

If this curve is homogeneous, i.e. $m = m_1 = \dots = m_s$, then we claim first that $\mu = \mu(\mathcal{O}_{\mathbb{P}^2}(1); \mathbb{E}) = m/d$. Indeed, this is an effective divisor on the ray R and it is not big (because big divisors on surfaces intersect all nef divisors positively, see [6, Corollary 3.3]), so it must be the point where the ray leaves the big cone.

Now, suppose that for some $\varepsilon < \delta < \mu$ the ray R crosses another Zariski chamber wall. This means that there is a divisor $eH - k\mathbb{E}$ (obtained after possible symmetrization of a curve D with 0 intersection number with $H - \delta\mathbb{E}$) with

$$(H - \delta\mathbb{E}) \cdot (eH - k\mathbb{E}) = 0.$$

Hence $e = \delta ks < \mu ks$. On the other hand

$$(eH - k\mathbb{E}) \cdot (dH - \mu\mathbb{E}) = e - k\mu s < 0,$$

implies that C is a component $eH - k\mathbb{E}$ which is not possible. Hence there are only two Zariski chambers meeting the ray R in this case.

If the curve C is not homogeneous, then since the points are general there exist at least (and also at most) s different irreducible curves computing ε . All these curves are in the support of the negative part of the Zariski decomposition of $H - \lambda\mathbb{E}$ for $\lambda > \varepsilon$. Hence their intersection matrix is negative definite and this is a matrix of maximal dimension (namely s) with this property. This implies that R cannot meet another Zariski chamber because the support of the negative part of Zariski decompositions grows only when encountering new chambers, see [5]. \square

It is interesting to compare this result with the following easy example, which constructs rays meeting a maximal number of chambers.

Example 2.5. Keeping the notation from Proposition 2.4 let $L = (\frac{s(s+1)}{2} + 1)H - E_1 - 2E_2 - \dots - sE_s$ is an ample divisor on X and the ray $R = L + \lambda\mathbb{E}$ crosses $s+1 = \rho(X)$ Zariski chambers. Indeed, with λ growing, exceptional divisors E_1, E_2, \dots, E_s join the support of the Zariski decomposition of $L - \lambda\mathbb{E}$ one by one. We leave the details to the reader.

We conclude this section with the proof of Lemma 2.2.

Proof of Lemma 2.2. Since $f(1/\sqrt{s+1}) = 0$ it is enough to show that $f(\delta)$ is increasing for $1/\sqrt{s+1} \leq \delta \leq 1/\sqrt{s}$. Consider the derivative

$$f'(\delta) = s \left(1 + \frac{\delta}{\sqrt{1-s\delta^2}}(s - 2\sqrt{s+1}) - \sqrt{s+1} \right). \quad (9)$$

The function $h(\delta) = \frac{\delta}{\sqrt{1-s\delta^2}}$ is increasing for $1/\sqrt{s+1} \leq \delta \leq 1/\sqrt{s}$ since the numerator is an increasing function of δ and the denominator is a decreasing function of δ . We have $h(\frac{1}{\sqrt{s+1}}) = 1$ so that $h(\delta) \geq 1$ holds for all δ . Since the coefficient at $h(\delta)$ in (9) is positive we have

$$f'(\delta) \geq s(1 + (s - 2\sqrt{s+1}) - \sqrt{s+1}) = 1 + s - 3\sqrt{s+1} > 0,$$

which completes the proof. \square

3 Rationality of Seshadri constants and functions on Okounkov bodies

The theory of Newton–Okounkov bodies has emerged recently with work by Okounkov [21], Kaveh–Khovanskii [14], and Lazarsfeld–Mustață [18]. Shortly thereafter, Boucksom–Chen [7] and Witt–Nyström [20] have shown ways of constructing geometrically significant functions on Okounkov bodies, that were further studied

in [16]. In the context of this note the study of Okounkov functions was pursued by the last three authors in [16]. We refer to [16] for construction and properties of Okounkov functions.

In this section we consider an arbitrary smooth projective surface X and an ample line bundle L on X . Let $p \in X$ be an arbitrary point and let $\pi : Y \rightarrow X$ be the blow up of p with exceptional divisor E . Recall that the Seshadri constant of L at p can equivalently be defined as

$$\varepsilon(L; p) = \sup \{ t > 0 \mid \pi^*L - tE \text{ is nef} \} .$$

There is a related invariant

$$\mu(L; p) \stackrel{\text{def}}{=} \sup \{ t > 0 \mid \pi^*L - tE \text{ is pseudo-effective} \} = \sup \{ t > 0 \mid \pi^*L - tE \text{ is big} \} .$$

The invariant $\varepsilon(L; p)$ is the value of the parameter λ where the ray $\pi^*L - \lambda E$ meets the boundary of the nef cone of Y , and $\mu(L; p)$ is the value of λ where the ray meets the boundary of the pseudo-effective cone. The following relation between the two invariants is important in our considerations.

Remark 3.1. If $\varepsilon(L; p)$ is irrational, then

$$\varepsilon(L; p) = \mu(L; p) .$$

In particular, if $\mu(L; p)$ is rational, then so is $\varepsilon(L; p)$.

Rationality of $\mu(L; p)$ implies rationality of the associated Seshadri constants on surfaces. This invariant appears in the study of the concave function φ_{ord_p} associated to the geometric valuation on X defined by the order of vanishing ord_p at p . We fix some flag $Y_\bullet : X \supseteq C \supseteq \{x_0\}$ and consider the Okounkov body $\Delta_{Y_\bullet}(L)$ defined with respect to that flag. We define also a multiplicative filtration determined by the geometrical valuation ord_P on the graded algebra $V = \bigoplus_{k \geq 0} V_k$ with $V_k = H^0(X, kL)$ by

$$\mathcal{F}_t(V) = \{ s \in V : \text{ord}_P(s) \geq t \} ,$$

see [16, Example 3.7] for details. (All the above remains valid in the more general context of graded linear series.) There is an induced filtration $\mathcal{F}_\bullet(V_k)$ on every summand of V and one defines the maximal jumping numbers of both filtrations as

$$e_{\max}(V, \mathcal{F}_\bullet) = \sup \{ t \in \mathbb{R} : \exists k \mathcal{F}_{kt} V_k \neq 0 \} \quad \text{and} \quad e_{\max}(V_k, \mathcal{F}_\bullet) = \sup \{ t \in \mathbb{R} : \mathcal{F}_t V_k \neq 0 \}$$

respectively. Let $\varphi_{\text{ord}_P}(x) = \varphi_{\mathcal{F}_\bullet}(x)$ be the Okounkov function on $\Delta_{Y_\bullet}(L)$ determined by filtration \mathcal{F}_\bullet , see [16, Definition 4.8]. It turns out that $\mu(L; p)$ is the maximum of the Okounkov function φ_{ord_P} .

Proposition 3.2. *With notation as above we have that*

$$\mu(L; p) = \limsup_{m \rightarrow \infty} \frac{\max \{ \text{ord}_p(s) \mid s \in H^0(X, \mathcal{O}_X(mL)) \}}{m} = \max_{x \in \Delta_{Y_\bullet}(L)} \phi_{\text{ord}_p}(x) .$$

Proof. Observe that

$$\text{ord}_p(s) = \text{ord}_E(\pi^*s) = \max \{ m \in \mathbb{N} \mid \text{div}(\pi^*s) - mE \text{ is effective} \} .$$

Consequently,

$$\begin{aligned} \mu(L; p) &= \sup \{ t \in \mathbb{R}_{\geq 0} \mid \pi^*L - tE \text{ is pseudo-effective} \} \\ &= \limsup_{m \rightarrow \infty} \frac{\max \{ \text{ord}_p(s) \mid s \in H^0(X, \mathcal{O}_X(mL)) \}}{m} , \end{aligned}$$

which gives the first equality.

For the second equality, we observe first that

$$\max \{ \text{ord}_p(s) \mid s \in H^0(X, \mathcal{O}_X(mL)) \} = e_{\max}(V_m, \mathcal{F}_\bullet) ,$$

and hence

$$\limsup_{m \rightarrow \infty} \frac{\max \{ \text{ord}_p(s) \mid s \in H^0(X, \mathcal{O}_X(mL)) \}}{m} = e_{\max}(V, \mathcal{F}_\bullet) .$$

Since

$$e_{\max}(V, \mathcal{F}_\bullet) = \max_{x \in \Delta_{Y_\bullet}(L)} \varphi_{\text{ord}_p}(x)$$

by Theorem 3.4, we are done. \square

3.1 Independence of the maximum of an Okounkov function on the flag

In the course of this section the projective variety X can have arbitrary dimension.

Boucksom and Chen proved that though $\varphi_{\mathcal{F}_\bullet}$ and $\Delta(V_\bullet)$ depend on the flag Y_\bullet , the integral of $\varphi_{\mathcal{F}_\bullet}$ over $\Delta(V_\bullet)$ is independent of Y_\bullet , [7, Remark 1.12 (ii)]. We prove now that the maximum of the Okounkov function does not depend on the flag. This fact is valid in the general setting of arbitrary multiplicative filtration \mathcal{F} defined on a graded linear series V_\bullet .

Remark 3.3. Note that in general the functions $\varphi_{\mathcal{F}_\bullet}$ are only upper-semicontinuous and concave, but not continuous on the whole Newton–Okounkov body as explained in [16, Theorem 1.1]. They are however continuous provided the underlying body $\Delta(V_\bullet)$ is a polytope (see again [16, Theorem 1.1]), which is the case for complete linear series on surfaces [15].

Theorem 3.4 (Maximum of Okounkov functions). *With the above notation, we have that*

$$\max_{x \in \Delta_{Y_\bullet}(L)} \varphi_{\mathcal{F}_\bullet}(x) = e_{\max}(L, \mathcal{F}_\bullet).$$

In particular the left hand side does not depend on the flag Y_\bullet .

Proof. For any real $t \geq 0$, we consider the partial Okounkov body $\Delta_{t, Y_\bullet}(L)$ associated the graded linear series $V_{t, k} \subset H^0(kL)$ given by

$$V_{t, k} \stackrel{\text{def}}{=} \mathcal{F}_{kt}(H^0(kL)) .$$

Note that by definition

$$e_{\max}(L, \mathcal{F}_\bullet) = \sup \{ t \in \mathbb{R} \mid \cup_k V_{t, k} \neq \emptyset \}$$

In other words,

$$e_{\max}(L, \mathcal{F}_\bullet) = \sup \{ t \in \mathbb{R} \mid \Delta_{t, Y_\bullet}(L) \neq \emptyset \}.$$

Recall that by definition

$$\varphi_{\mathcal{F}_\bullet}(x) = \sup \{ t \in \mathbb{R} \mid x \in \Delta_{t, Y_\bullet}(L) \}.$$

and it is therefore immediate that $\forall x$

$$\varphi_{\mathcal{F}_\bullet}(x) \leq e_{\max}(L, \mathcal{F}_\bullet).$$

from which it follows that

$$\max_{x \in \Delta_{Y_\bullet}(L)} \varphi_{\mathcal{F}_\bullet}(x) \leq e_{\max}(L, \mathcal{F}_\bullet).$$

Since the bodies $\Delta_{t,Y_\bullet}(L)$ form a decreasing family of closed subsets of \mathbb{R}^d , we have that

$$\cap_{t | \Delta_{t,Y_\bullet}(L) \neq \emptyset} \Delta_{t,Y_\bullet}(L) \neq \emptyset.$$

Consider a point $y \in \cap_{t | \Delta_{t,Y_\bullet}(L) \neq \emptyset} \Delta_{t,Y_\bullet}(L)$. We then have that

$$y \in \Delta_{t,Y_\bullet}(L) \Leftrightarrow \Delta_{t,Y_\bullet}(L) \neq \emptyset$$

and hence

$$\sup\{t \in \mathbb{R} | y \in \Delta_{t,Y_\bullet}(L)\} = \sup\{t \in \mathbb{R} | \Delta_{t,Y_\bullet}(L) \neq \emptyset\}$$

or in other words

$$\varphi_{\mathcal{F}_\bullet}(y) = e_{\max}(L, \mathcal{F}_\bullet)$$

from which it follows that

$$\max_{x \in \Delta_{Y_\bullet}(L)} \varphi_{\mathcal{F}_\bullet}(x) \leq e_{\max}(L, \mathcal{F}_\bullet).$$

This completes the proof of the theorem. \square

4 The effect of blowing up on Okounkov bodies and functions

We begin with an observation (valid in fact in arbitrary dimension, though we state and prove it here only for surfaces.)

Proposition 4.1. *Let S be an arbitrary surface with a fixed flag Y_\bullet and let $f : X \rightarrow S$ be the blow up of S at a point P not contained in the divisorial part of the flag, $P \notin Y_1$. Let E be the exceptional divisor. And finally let D be a big divisor on S . For any rational number λ such that $0 \leq \lambda < \mu(L; P)$ we let D_λ be the \mathbb{Q} -divisor $f^*D - \lambda E$. There is then a natural inclusion*

$$\Delta_{Y_\bullet}(D_\lambda) \subset \Delta_{Y_\bullet}(D).$$

Moreover, a filtration \mathcal{F}_\bullet on the graded algebra $\oplus_{k \geq 0} H^0(S; kD)$ induces a filtration $\mathcal{F}_\bullet^\lambda$ on the graded (sub)algebra $\oplus H^0(X, kD_\lambda)$, where the sum is taken over all k divisible enough. For associated Okounkov functions we have

$$\varphi_{\mathcal{F}_\bullet^\lambda}(x) \leq \varphi_{\mathcal{F}_\bullet}(x) \tag{10}$$

for all $x \in \Delta_{Y_\bullet}(D_\lambda)$.

Remark 4.2. The best case scenario is that the functions ϕ are piecewise linear with rational coefficients over a rational polytope. Of these properties, some evidence for the first was given by Donaldson [11] in the toric situation. For the second condition, it was proven in [1] that every line bundle on a surface has an Okounkov body which is a rational polytope.

Proof. Note first that since the blow up center is disjoint from all elements in the flag, one can take Y_\bullet to be an admissible flag on X . (Strictly speaking one takes f^*Y_\bullet as the flag, but it should cause no confusion to identify flag elements upstairs and downstairs.)

Then, if k is sufficiently divisible we have

$$H^0(X, f^*kD - k\lambda E) = \{s \in H^0(S, kD) : \text{ord}_P(s) \geq E\} \subset H^0(S, kD).$$

The inclusion of the Okounkov bodies follows immediately under this identification. We can view the algebra associated to $f^*D - E$ as a graded linear series on S .

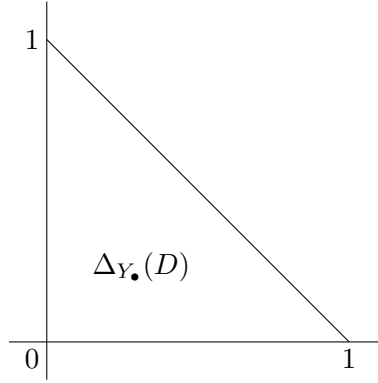
The claim about the Okounkov functions follows from their definition, see [16, Definition 4.8]. Indeed, the supremum arising in the definition of $\varphi_{\mathcal{F}_\bullet}$ is taken over a smaller set of sections than it is for $\varphi_{\mathcal{F}_\bullet}$. \square

The following examples illustrate various situations arising in the setting of Proposition 4.1.

Example 4.3. Let ℓ be a line in $X_0 = \mathbb{P}^2$ and let $P_0 \in \ell$ be a point. We fix the flag

$$Y_\bullet : X_0 \supset \ell \supset \{P_0\}.$$

Let $D = \mathcal{O}_{\mathbb{P}^2}(1)$. Then $\Delta_{Y_\bullet}(D)$ is simply the standard simplex in \mathbb{R}^2 .



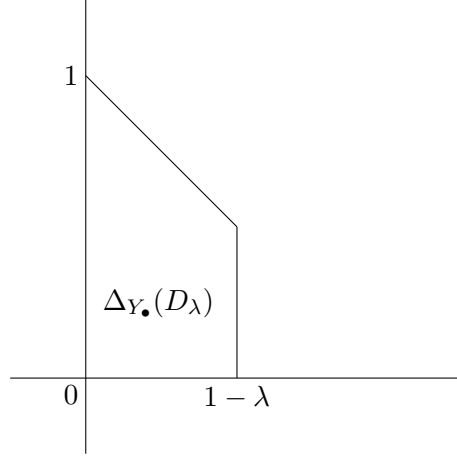
Let \mathcal{F}_\bullet be the filtration on the complete linear series of D imposed by the geometric valuation $\nu = \text{ord}_{P_0}$ and let φ_ν be the associated Okounkov function. Then

$$\varphi_\nu(a, b) = a + b.$$

Indeed, given a point (a, b) with rational coordinates, we pass to the integral point (ka, kb) . This valuation vector can be realized geometrically by a global section in $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k))$ vanishing exactly with multiplicity ka along ℓ , exactly with multiplicity kb along a line passing through P_0 different from ℓ and along a curve of degree $k(1 - a - b)$ not passing through P_0 .

The next example shows that even when the Okounkov body changes in the course of blowing up, the Okounkov function may remain the same.

Example 4.4. Keeping the notation from the previous Example and from Proposition 4.1 let $f : X_1 = \text{Bl}_{P_1} \mathbb{P}^2 \rightarrow X_0 = \mathbb{P}^2$ be the blow up of the projective plane in a point P_1 not contained in the flag line ℓ with the exceptional divisor E_1 . We work now with a \mathbb{Q} -divisor $D_\lambda = f^*(\mathcal{O}_{\mathbb{P}^2}(1)) - \lambda E_1 = H - \lambda E_1$, for some fixed $\lambda \in [0, 1]$. A direct computation using [18, Theorem 6.2] gives that the Okounkov body has the shape



Thus we see that the Okounkov body of D_{λ} is obtained from that of D by intersecting with a closed halfspace.

For the valuation $\nu = \text{ord}_{P_0}$, we get as above

$$\varphi_{\nu}(a, b) = a + b.$$

Let now k be an integer such that the point (ka, kb) is integral and $k\lambda$ is also an integer. Now we need to exhibit a section s in $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(k))$ satisfying the following conditions:

- a) s vanishes along ℓ exactly to order a ;
- b) s vanishes in the point P_1 to order at least $k\lambda$;
- c) s vanishes in the point P_0 exactly to order b .

We let the divisor of s to consist of a copies of ℓ (there is no other choice here), of b copies of the line through P_0 and P_1 , of $k\lambda - b$ copies of any other line passing through P_1 (if this number is negative then this condition is empty) and of a curve of degree $k(1 - a - \max\{b, \lambda\})$ passing neither through P_0 , nor through P_1 .

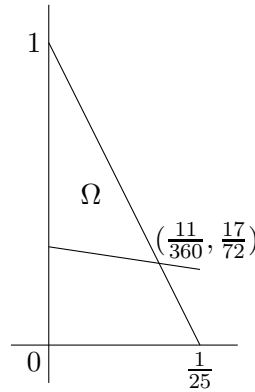
Remark 4.5. Note that in the setting of Proposition 4.1 Okounkov bodies of divisors D_{λ} always result from those of D by cutting with finitely many halfplanes. This is an immediate consequence of [15, Theorem 5].

We conclude by showing that the inequality in (10) can be sharp, i.e. the blow up process can influence the Okounkov function as well as the Okounkov body.

Example 4.6. Keeping the notation from the previous examples, let $f : X_6 \rightarrow \mathbb{P}^2$ be the blow up of six general points P_1, \dots, P_6 not contained in ℓ and chosen so that the points P_0, P_1, \dots, P_6 are also general. Let E_1, \dots, E_6 denote the exceptional divisors and set $\mathbb{E} = E_1 + \dots + E_6$. We consider the divisor $D = H - \frac{2}{5}\mathbb{E}$. A direct computation using [18, Theorem 6.2] (this requires computing Zariski decompositions this time, see [2] for an effective approach) yields the triangle with vertices at the origin and in points $(0, 1)$ and $(1/25, 0)$ as $\Delta_{Y_{\bullet}}(D)$. For the valuation $\nu = \text{ord}_{P_0}$ we get now

$$\varphi_{\nu}(a, b) \leq 4/15 < a + b \tag{11}$$

for $(a, b) \in \Omega = \{(x, y) \in \mathbb{R}^2 : x \in [0, 11/360] \text{ and } b \in (4/15 - a, 1 - 25a]\}$.



The reason for the above inequality is the following. Let $(a, b) \in \Omega$ be a valuation vector. Assume to the contrary that $\varphi(a, b) > 4/15$. It is well known that $\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1), P_0, \dots, P_6) = \frac{3}{8}$, see for instance [23, Example 2.4]. On the other hand a section with the above valuation vector would have (after scaling to $\mathcal{O}(1)$) multiplicities $2/5$ at P_1, \dots, P_6 and $\varphi_\nu(a, b) > 4/15$ at P_0 . It would give Seshadri quotient

$$\frac{1}{6 \cdot \frac{2}{5} + \varphi_\nu(a, b)} < \frac{3}{8},$$

a contradiction. This proves (11).

Since the SHGH Conjecture holds for 9 points, the first challenge arising in the view of our Theorem would be to compute the Okounkov body and the Okounkov function associated to ord_{P_0} as above for the system

$$22H - 7(E_1 + \dots + E_9).$$

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